Conductivity of the ionosphere

Stefan Maus, CIRES, University of Colorado, January 19, 2006

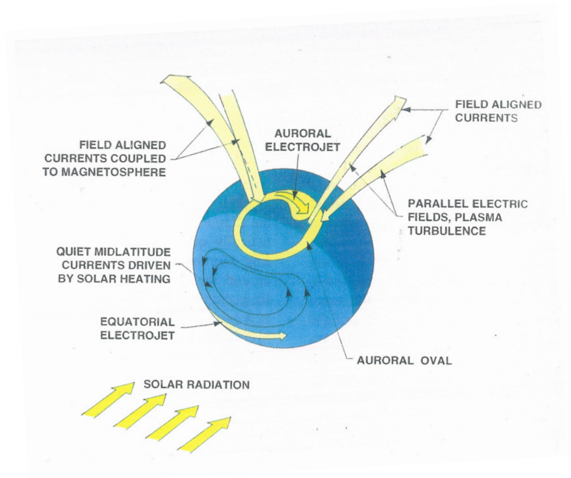


Figure 1: Ionospheric current systems (Langel, 1989)

1 The ionosphere

Atmospheric composition

- Neutral atoms and molecules (neutral gas)
- Free ions and free electrons (ionospheric plasma)

The ionospheric plasma is generated through dissociation by

- Particle precipitation (at high latitudes)
- Energetic ultraviolet (EUV) solar radiation (all latitudes)

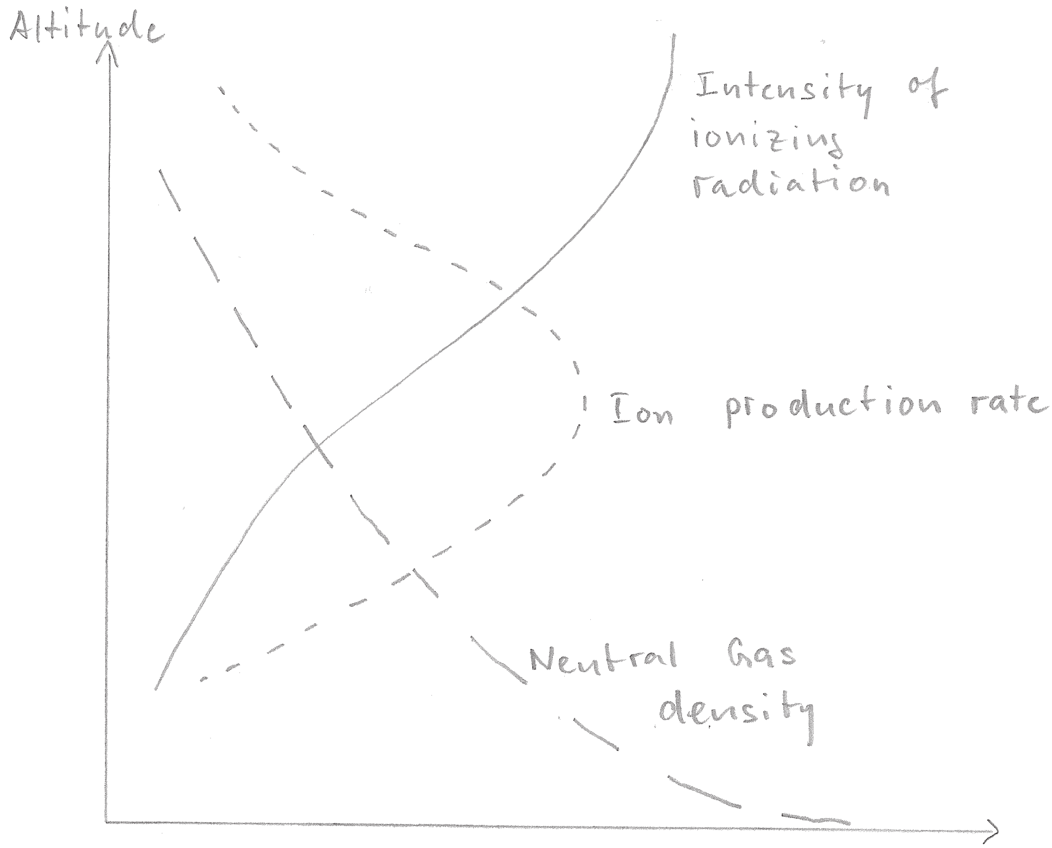


Figure 2: The ion production rate with height is a function of the intensity of the ionizing radiation as well as the density of the neutral gas

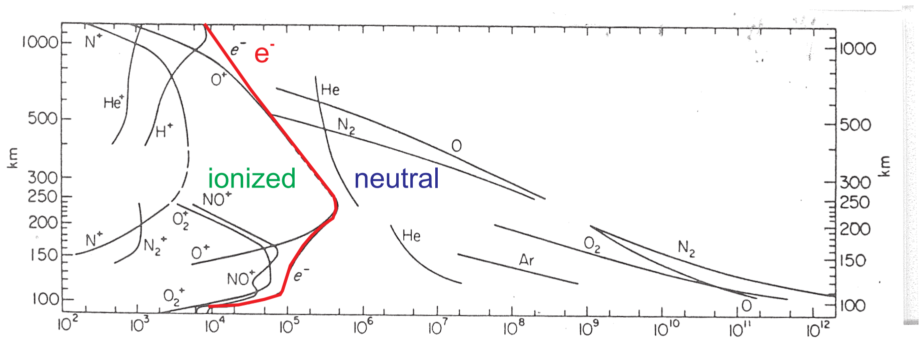


Figure 3: Day side vertical profile of atmospheric composition over New Mexico, from rocket and satellite measurements (Kelly, 1989)

For the following derivations, we can safely assume that

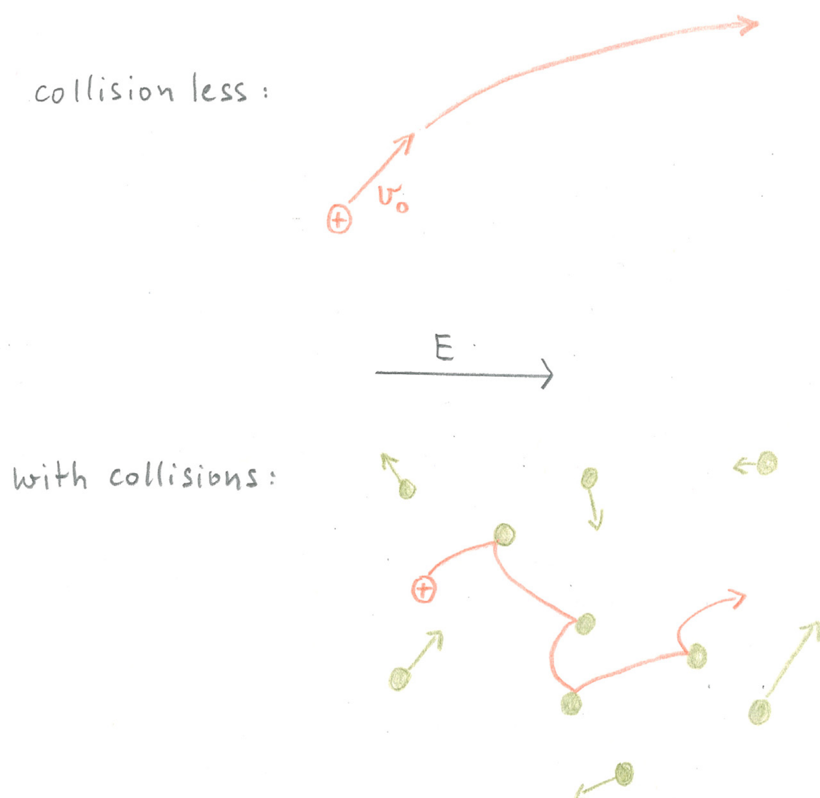
- the contribution of ions with a charge higher than +1 is negligible
- the number density of ions n_i is equal to the number density of electrons n_e
- in the range of the atmosphere considered here (50-1000 km), the neutral gas density significantly exceeds the plasma density. Thus, the gas is weakly ionized.

2 Electrical conductivity in weakly ionized gas

The local coordinate system is chosen to move with the neutral gas. Thus, the neutral gas is statistically at rest in this system, although of course, the individual particles move at their thermal velocity. First, consider the case in which there is an electric field but no magnetic field.

2.1 Conductivity in the absence of a magnetic field

For an ion ⊕ we then have:



In addition to the velocity which the particle acquires in collisions, it is accelerated in the electric field by $\mathbf{a} = q\mathbf{E}/m$. To infer the corresponding electric current vector, we need to calculate the mean velocity vector of the charged particle.
Let us assume that in every collision the particle loses the additional velocity acquired in the electric field.
During the time τ_1 between the first and the second collision, the particle acquires the additional velocity $\mathbf{v}(\tau_1) = \mathbf{a}\tau_1$.
On average, the velocity in this time interval is

$$\bar{\mathbf{v}}_\mathbf{1} = \frac{1}{\tau_1}\int_0^{\tau_1} \mathbf{a}t\; dt = \frac{1}{2}\mathbf{a}\tau_1 \tag{1}$$

Similarly, in the time interval τ_2 from the 2nd to the 3rd collision $\bar{\mathbf{v}}_\mathbf{2} = \frac{1}{2}\mathbf{a}\tau_2$
The average interval duration is $\bar{\tau} = \frac{1}{2}(\tau_1 + \tau_2)$.
Now we can compute the average velocity for the two intervals:

$$\begin{aligned} \bar{\mathbf{v}}_{\mathbf{1},\mathbf{2}} &= \frac{\tau_1\bar{\mathbf{v}}_\mathbf{1} + \tau_2\bar{\mathbf{v}}_\mathbf{2}}{\tau_1 + \tau_2} && (2) \\ &= \underbrace{\frac{1}{2}}_{p(\tau_1)}\frac{\tau_1}{\bar{\tau}}\bar{\mathbf{v}}_\mathbf{1} + \underbrace{\frac{1}{2}}_{p(\tau_2)}\frac{\tau_2}{\bar{\tau}}\bar{\mathbf{v}}_\mathbf{2} && (3) \end{aligned}$$

At this point we need a realistic assumption on the probability distribution $p(\tau)$ of the time interval between two collisions having a length of τ. Because we can then infer the mean velocity of a particle as

$$\begin{aligned} \bar{\mathbf{v}} &= \int_0^\infty p(\tau)\,\frac{\tau}{\bar{\tau}}\,\frac{1}{2}\mathbf{a}\tau\; d\tau && (4) \\ &= \frac{a}{2\bar{\tau}}\int_0^\infty \tau^2 p(\tau)\; d\tau && (5) \end{aligned}$$

Let us assume that, at any point in time, the time to the next collision does not depend on the history of the particle. This is actually quite a strong assumption because the particle is being accelerated in the electric field. And the faster the particle the earlier we expect the next collision. Therefore, we have to assume that $v_E \ll v_{thermal}$. In analogy to the process of radioactive decay, in which the time to the next decay does not depend on the time passed since the previous decay, we can assume an exponential probability distribution

$$p(\tau) = \frac{1}{\bar{\tau}}\exp\frac{-\tau}{\bar{\tau}} \tag{6}$$

The expectance for the collisionless time interval (or simply its “mean”) is then

$$E[\tau] = \int_0^\infty \tau p(\tau)\; d\tau = \bar{\tau} \tag{7}$$

Using this probability distribution in (5) gives

$$\begin{aligned} \bar{\mathbf{v}} &= \frac{\mathbf{a}}{2\bar{\tau}}\int_0^\infty \tau^2\frac{1}{\bar{\tau}}\exp\frac{-\tau}{\bar{\tau}}\; d\tau && (8) \\ &= \frac{\mathbf{a}}{2\bar{\tau}}2\bar{\tau}^2 && (9) \\ &= \mathbf{a}\bar{\tau} && (10) \\ \Rightarrow\quad & \bar{\mathbf{v}} = \frac{q\mathbf{E}}{m}\bar{\tau} && (11) \end{aligned}$$

This is the mean velocity of a single particle. Assuming that this mean velocity is representative for all of the charged particles of a particular kind, we only need the ion number density n_i and the electron number density n_e to get the current density vector $\mathbf{j}$ as

$$\begin{aligned} \mathbf{j} &= n_i \bar{\mathbf{v}}_{\mathbf{i}} q_i + n_e \bar{\mathbf{v}}_{\mathbf{e}} q_e && (12) \\ &= ne^2 \left(\frac{\bar{\tau}_i}{m_i} + \frac{\bar{\tau}_e}{m_e} \right) \mathbf{E} && (13) \end{aligned}$$

Now we substitute $\bar{\tau}$ with the collision frequency $\nu = 1/\bar{\tau}$ and obtain

$$\mathbf{j} = \underbrace{ne^2 \left(\frac{1}{m_i \nu_i} + \frac{1}{m_e \nu_e} \right)}_{\sigma_0} \mathbf{E} \tag{14}$$

If there is more than one kind of ions, one has to adjust the equation accordingly.

2.2 Conductivity in presence of a magnetic field

In the presence of a magnetic field it is advantageous to split the electric field vector into the component $\mathbf{E_{par}}$ parallel to the magnetic field and the component $\mathbf{E_{perp}}$ perpendicular to $\mathbf{B}$.

2.2.1 Parallel to magnetic field

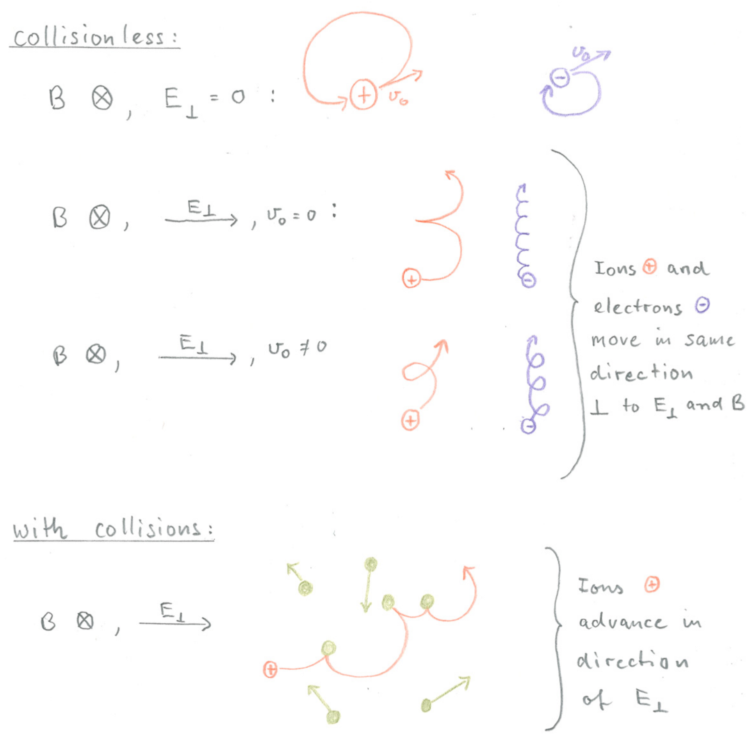
Since no Lorentz forces act on a particle moving parallel to the magnetic field

$$\mathbf{j}_{||} = \sigma_0 \mathbf{E}_{||} \tag{15}$$

for this reason, σ_0 is also called the "parallel" conductivity.

2.2.2 Conductivity perpendicular to the magnetic field

Consider again the path of a particle:



The charged particle now experiences two contributions to its acceleration:

$$\mathbf{a} = \frac{q\mathbf{E}_\perp}{m} + \underbrace{\frac{q}{m}(\mathbf{v} \times \mathbf{B})}_{Lorentz\ acceleration} \tag{16}$$

We choose a coordinate system with the unit vectors

$$\hat{\mathbf{z}} = \frac{\mathbf{B}}{|B|},\ \hat{\mathbf{x}} = \frac{\mathbf{E}_\perp}{|E_\perp|},\ \hat{\mathbf{y}} = -\frac{\mathbf{v} \times \mathbf{B}}{|\mathbf{v} \times \mathbf{B}|} \tag{17}$$

We are only interested in the particle movement in the (x,y) plane.

$$a_x = \frac{q}{m}E_\perp + \frac{q}{m}v_y B \tag{18}$$

$$a_y = -\frac{q}{m}v_x B \tag{19}$$

where $E_\perp = |\mathbf{E}_\perp|$ and $B = |\mathbf{B}|$.
We write $\omega = qB/m$ Note that this ω carries the sign of the charge q, while the gyro-frequency $|\omega|$ is always positive.
If we further define $a_\perp = \frac{q}{m}E_\perp$

$$a_x = a_\perp + \omega v_y \tag{20}$$

$$a_y = -\omega v_x \tag{21}$$

The acceleration is the temporal derivative of the velocity, so

$$\dot{v}_x = a_\perp + \omega v_y \tag{22}$$

$$\dot{v}_y = -\omega v_x \tag{23}$$

$$\ddot{v}_x = -\omega^2 v_x \rightarrow v_x = C\sin(\omega t + \phi) \tag{24}$$

$$v_y = \frac{\dot{u}_x - a_\perp}{\omega} = C\cos(\omega t + \phi) - \frac{a_\perp}{\omega} \tag{25}$$

The constants C and ϕ follow from the initial velocity of a particle after a collision. However, inserting $\mathbf{v} = \mathbf{v}' + \mathbf{v_0}$ into (16), we see that the initial velocity $\mathbf{v_0}$ only leads to an additional gyration which cancels out in the average over all collisions. Therefore, we can set $v_x(0) = 0$ and $v_y(0) = 0$ and then obtain $\phi = 0$ and $C = a_\perp/\omega$. Then the particle trajectories are

$$v_x = \frac{a_\perp}{\omega}\sin\omega t \tag{26}$$

$$v_y = \frac{a_\perp}{\omega}(\cos\omega t - 1) \tag{27}$$

Now we can average the velocity in a single time interval between two collisions.

$$\bar{v}_x(\tau) = \frac{1}{\tau}\int_0^\tau v_x(t)\,dt \tag{28}$$

$$= \frac{a_\perp}{\tau\omega}\int_0^\tau \sin\omega t\,dt \tag{29}$$

$$= \frac{a_\perp}{\tau\omega^2}(1 - \cos\omega\tau) \tag{30}$$

And then average over all possible collisionless intervals, given the probability distribution $p(\tau)$, and obtain

$$\begin{aligned}
\bar{v}_x &= \int_0^\infty p(\tau)\frac{\tau}{\bar{\tau}}\bar{v}_x(\tau)\,d\tau && (31)\\
&= \frac{1}{\bar{\tau}}\int_0^\infty p(\tau)\frac{a_\perp}{\omega}(1-\cos\omega\tau)\,d\tau && (32)\\
&= \frac{a_\perp}{\bar{\tau}\omega}\left(\int_0^\infty p(\tau)\,d\tau - \int_0^\infty p(\tau)\cos\omega\tau\,d\tau\right) && (33)\\
&= \frac{a_\perp}{\bar{\tau}\omega}\left(1-\frac{1}{1+\bar{\tau}^2\omega^2}\right) && (34)\\
&= \frac{a_\perp\bar{\tau}}{1+\bar{\tau}^2\omega^2} && (35)
\end{aligned}$$

Substituting $\bar{\tau} = 1/\nu$ and $a_\perp = qE_\perp/m$ then gives

$$\bar{v}_x = \frac{q\nu}{m(\nu^2+\omega^2)}E_\perp \qquad (36)$$

From the mean particle velocity follows the current density

$$\begin{aligned}
j_x &= n(\bar{v}_{x,i}q_i + \bar{v}_{x,e}q_e) && (37)\\
&= \underbrace{ne^2\left(\frac{\nu_i}{m_i(\nu_i^2+\omega_i^2)} + \frac{\nu_e}{m_e(\nu_e^2+\omega_e^2)}\right)}_{Pedersen\ conductivity\ \sigma_P} E_\perp && (38)
\end{aligned}$$

with the Pedersen conductivity giving the conductivity parallel to that component of the electric field which is perpendicular to the magnetic field.
Similarly, the conductivity perpendicular to $\mathbf{B}$ and perpendicular to $\mathbf{E}_\perp$ is obtained by first calculating by the mean particle velocity

$$\bar{v}_y = \frac{q}{m}\left(\frac{-\omega}{\nu^2+\omega^2}\right)E_\perp \qquad (39)$$

Then it is important to consider that $q_i = e$ while $q_e = -e$. Because of $\omega = qB/m$, the gyro-frequency of ions is

$$\omega_{g,i} = |\omega_i| = \omega_i \qquad (40)$$

while the gyro-frequency of electrons is

$$\omega_{g,e} = |\omega_e| = -\omega_e \qquad (41)$$

Then

$$\begin{aligned}
j_y &= n(\bar{v}_{y,i}q_i + \bar{v}_{y,e}q_e) && (42)\\
&= ne^2\left(\frac{-\omega_i}{m_i(\nu_i^2+\omega_i^2)} + \frac{-\omega_e}{m_i(\nu_e^2+\omega_e^2)}\right)E_\perp && (43)\\
&= ne^2\left(\frac{-\omega_i}{m_i(\nu_i^2+\omega_i^2)} + \frac{|\omega_e|}{m_i(\nu_e^2+\omega_e^2)}\right)E_\perp && (44)\\
&= \underbrace{ne^2\left(\frac{|\omega_e|}{m_i(\nu_e^2+\omega_e^2)} - \frac{\omega_i}{m_i(\nu_i^2+\omega_i^2)}\right)}_{Hall\ conductivity\ \sigma_H} E_\perp && (45)
\end{aligned}$$

2.3 Conductivity tensor

In summary, Ohm's law for the ionosphere can be written using the anisotropic 3D conductivity tensor $\underline{\boldsymbol{\sigma}}$ as:

$$\mathbf{j} = \underline{\boldsymbol{\sigma}}\mathbf{E}, \; where \; \underline{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_P & -\sigma_H & 0 \\ \sigma_H & \sigma_P & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix} \tag{46}$$

Here, the coordinate system is chosen with $\hat{\mathbf{z}}$ in the direction of the magnetic field, $\hat{\mathbf{x}}$ parallel to $\mathbf{E}_\perp$, and $\hat{\mathbf{y}}$ perpendicular to **B** and perpendicular to $\mathbf{E}_\perp$.

2.4 Underlying assumptions

In the preceding derivations, we used some implicit and explicit assumptions:

1. That the fields **B**, **E** and the particle densities n_i and n_e are locally homogeneous and constant
2. The assumption of an exponential probability density for the collisionless time interval τ is only realistic if the additional velocity caused by the electric field is small in comparison to the thermal velocities. Otherwise, one would have to assume an exponential distribution for the spatial, rather than for the temporal distance between two collisions.

2.5 Conductivity and altitude

The conductivity parallel to the magnetic field increases strongly with altitude due to decreasing collisions with the neutral gas. The parallel conductivity is always much higher than the conductivity perpendicular to the magnetic field. Indeed, throughout the ionosphere the parallel conductivity is so high that the magnetic field lines can usually be considered as equi-potentials of the electric field.

The movement of a class of charged particles perpendicular to the magnetic fields depends on the ratio of the collision frequency ν to the gyro frequency ω_g. If $\nu > \omega_g$ then collisions prevent the particle from gyrating and the particles move in the direction of the electric field, as a Pedersen current. If, on the other hand, $\nu < \omega_g$ then the particles predominantly drift perpendicular to the electric field.

The different behavior of ions and electrons leads to drastic changes of the electrical properties of the ionosphere with altitude. Typically, the collision frequency strongly exceeds the gyro frequency for both ions and electrons up to a height of about 70 km. Up to this altitude the parallel and Pedersen conductivities dominate.

Above about 70 km the electrons start to gyrate and drift perpendicular to the electric field, while the ions still move in direction of the electric field. This difference in direction is the basis for the Hall conductivity. From a different point of view: a neutral wind at this altitude can drag ions along with it. The electrons collide with the neutral gas much more frequently, but they can still gyrate. Consequently, they will immediately change direction and cannot be dragged along by the neutral wind. This leads to a separation of charge similar to a dynamo.

Above about 130 km altitude the ions also start to gyrate and drift perpendicular to the magnetic field. Because the drift of electrons and ions in the same direction does not constitute an electric current, this leads to a sharp decrease in Hall conductivity. Occasional collisions still play a role above 130 km, leading to a significant Pedersen conductivity which decreases at further altitude.

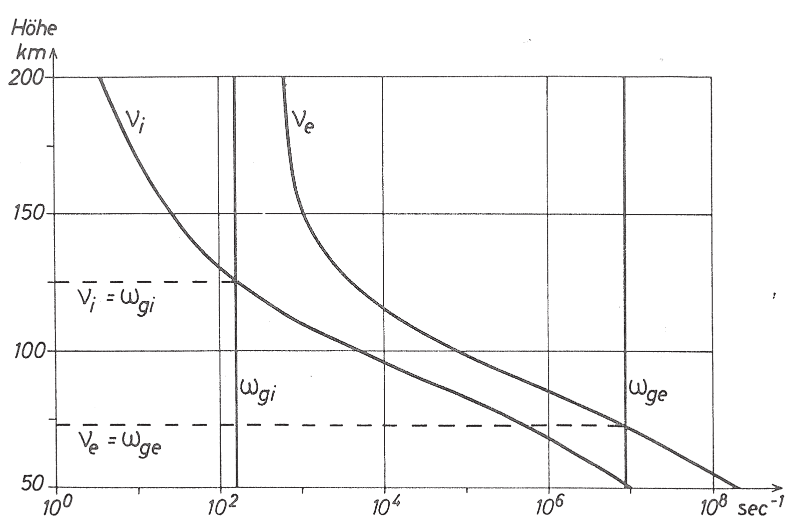


Figure 4: Day side vertical profile of collision frequencies of ions ν_i and electrons ν_e in comparison to their gyro-frequencies ω_{gi} and ω_{ge}. The dashed lines define the dynamo region (Kertz 1989).

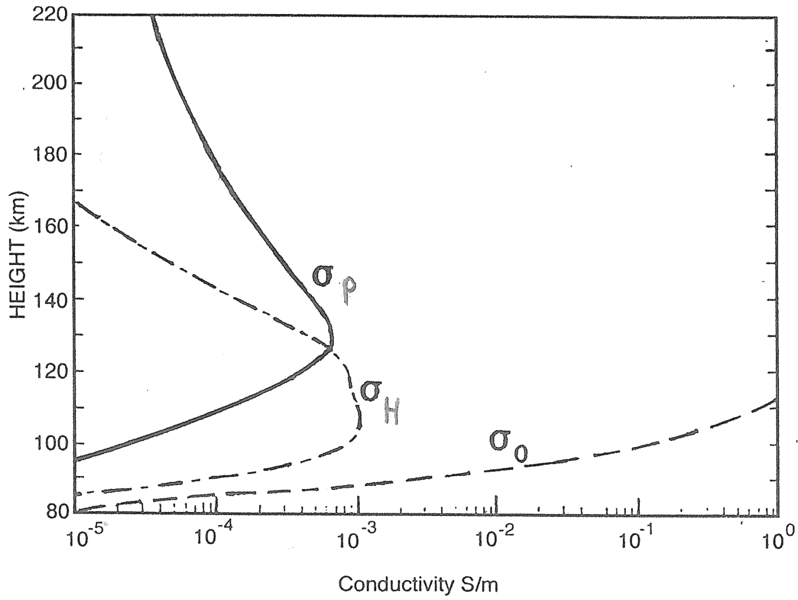


Figure 5: Day side vertical profile of the parallel, Pedersen and Hall conductivities (Campbell 1997)

3 Equatorial electrojet and Cowling conductivity

At the magnetic equator on the day side there is a strong electrical current which is normally directed eastward.

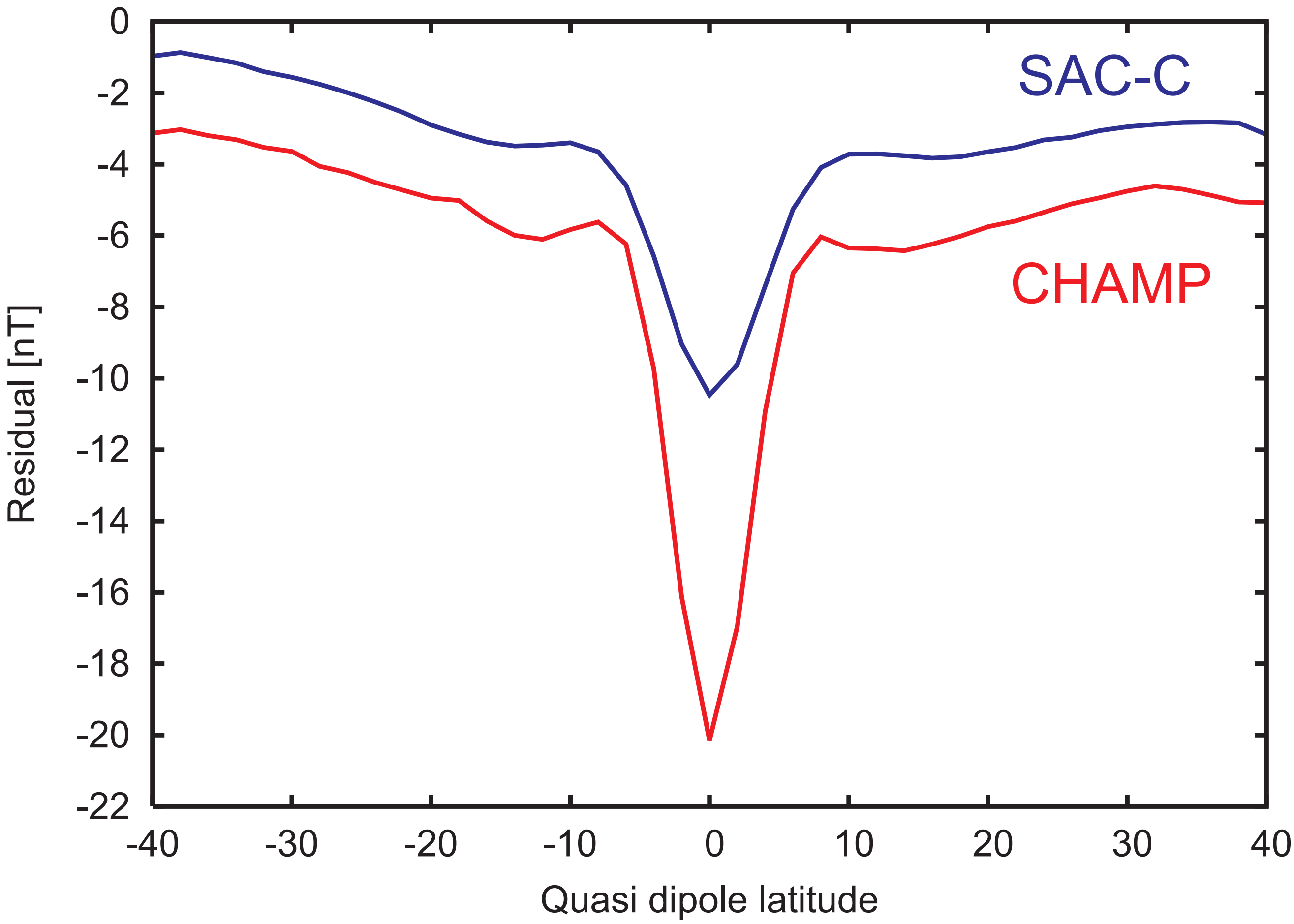


Figure 6: CHAMP (400 km altitude) and SAC-C (700 km altitude) satellite measurements of the magnetic field intensity at 10:00 to 11:00 local time, averaged over all longitudes.

This current has a simple explanation: We choose the right-handed geomagnetic coordinate system with x pointing north, y pointing east and z pointing down. An initial eastward electric field E_y generates a Hall current $j_z = \sigma_H E_y$. It generates a positive polarization charge on the underside of the E-region, and a negative charge on the top side:
This continues until the electrostatic secondary polarization field stops the vertical Hall current. What then happens can be seen as follows:

$$I \quad j_y = \sigma_P E_y - \sigma_H E_z \tag{47}$$
$$II \quad j_z = \sigma_P E_z + \sigma_H E_y \tag{48}$$

$$II \quad E_z = \frac{j_z}{\sigma_P} - \frac{\sigma_H}{\sigma_P} E_y \quad |j_z = 0 \tag{49}$$

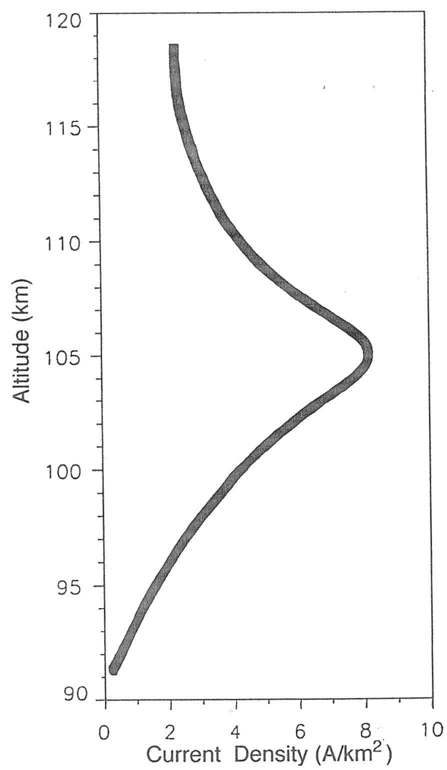
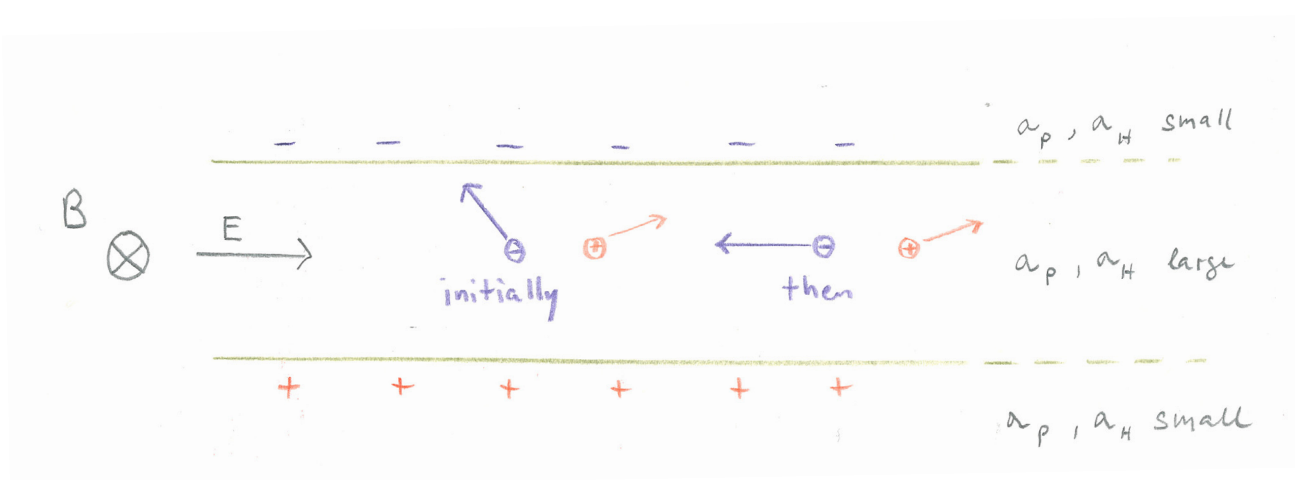


Figure 7: Rocket measurements of the vertical profile of the current density in the equatorial electrojet (Campbell 1997)



$$I \quad j_y = \sigma_P E_y + \sigma_H \frac{\sigma_H}{\sigma_P} E_y \tag{50}$$

$$= \underbrace{\sigma_P \left(1 + \frac{\sigma_H^2}{\sigma_P^2}\right)}_{Cowling\ conductivity\ \sigma_C} E_y \tag{51}$$

For a typical ratio of $\sigma_H/\sigma_P = 10$ we get $\sigma_C = 100\sigma_P$ which means that even a small eastward electric field can drive a strong eastward Cowling current, but only within a few degrees of the magnetic equator.

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